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R-deformed Heisenberg algebra

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Abstract

It is shown that the deformed Heisenberg algebra involving the reflection operator R (R-deformed Heisenberg algebra) has finite-dimensional representations which are equivalent to representations of paragrassmann algebra with a special differentiation operator. Guon-like form of the algebra, related to the generalized statistics, is found. Some applications of revealed representations of the R-deformed Heisenberg algebra are discussed in the context of OSp(2|2) supersymmetry. It is shown that these representations can be employed for realizing (2+1)-dimensional supersymmetry. They give also a possibility to construct a universal spinor set of linear differential equations describing either fractional spin fields (anyons) or ordinary integer and half-integer spin fields in 2+1 dimensions.

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1 Introduction

The deformed Heisenberg algebra involving the reflection operator R has found many interesting physical applications. It appeared naturally in the context of parafields [1, 2], but earlier it was known in connection with some quantum mechanical systems [3]. Recently this algebra was used for investigating quantum mechanical N-body Calogero model [4], for bosonization of supersymmetric quantum mechanics [5, 6, 7] and describing anyons in (2+1) [7, 8] and (1+1) dimensions [9]. In all the listed applications the infinite-dimensional unitary representations of the R-deformed Heisenberg algebra were used.

In the present paper it will be shown that this algebra has also finite dimensional representations which are equivalent to representations of some paragrassmann algebra [10] with differentiation operator realized in a special form. We shall show that guon-like algebra [11] can be constructed in a natural way proceeding from the R-deformed Heisenberg algebra. Such guon-like algebra can be related in some way to the q-deformed Heisenberg algebra [12, 13] with deformation parameter q being a primitive root of unity [10]. We shall discuss some applications of finite-dimensional representations of the R-deformed Heisenberg algebra. In particular, they will be used for realization of OSp(2|2) supersymmetry. The relationship of revealed representations to finite-dimensional representations of $(2+1)$-dimensional Lorentz group will be established. The latter will be employed for realizing (2+1)-dimensional supersymmetry. We shall also use them for constructing a universal spinor set of linear differential equations describing either fractional spin fields (anyons) or ordinary integer or half-integer spin fields in 2+1 dimensions.

2 Representations of R-deformed Heisenberg algebra

The R-deformed Heisenberg algebra is given by the generators a^-, a^+, 1 and by the reflection operator R satisfying the (anti)commutation relations [1]–[8]:

$$[a^-, a^+] = 1 + \nu R, \quad R^2 = 1, \quad \{a^\pm, R\} = 0, \tag{2.1}$$

and $[a^\pm, 1] = [R, 1] = 0$, where $\nu \in \mathbf{R}$ is a deformation parameter. The reflection operator R is hermitian, whereas a^+ and a^- will be considered as mutually conjugate operators with respect to appropriate scalar product. One introduces the vacuum state $|0\rangle$, $a^-|0\rangle = 0$, $\langle 0|0\rangle = 1$, $R|0\rangle = |0\rangle$, and defines the states $|n\rangle = C_n(a^+)^n|0\rangle$ with some normalization constants C_n. Then, from the relation

$$[a^-, (a^+)^n] = \left(n + \frac{1}{2}(1 - (-1)^n)\nu R\right)(a^+)^{n-1} \tag{2.2}$$

one concludes that algebra (2.1) has infinite-dimensional unitary representations when $\nu > -1$. In this case the states $|n\rangle$ with $C_n = ([n]_\nu!)^{-1/2}$, $[n]_\nu! = \prod_{l=1}^n [l]_\nu$, $[l]_\nu = l + \frac{1}{2}(1 - (-1)^l)\nu$, form the complete orthonormal basis of Fock space representation, $\langle n|n'\rangle = \delta_{nn'}$. The reflection operator can be realized in terms of creation and annihilation operators via the number operator [2, 7],

$$N = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}(\nu + 1), \quad N|n\rangle = n|n\rangle, \tag{2.3}$$

$$R = (-1)^N = \cos \pi N. \tag{2.4}$$

On the other hand, one can consider R-deformed Heisenberg algebra (2.1) working in the Schrödinger representation, $\Psi = \Psi(x)$, with creation-annihilation operators realized in the usual form $a^{\pm} = \frac{1}{\sqrt{2}}(x \mp ip)$. Here the deformed momentum operator is [1] $p = -i(\frac{d}{dx} - \frac{\nu}{2x}R)$, and operator R acts as $R\Psi(x) = \Psi(-x)$, and so, $R\Psi_{\pm}(x) = \pm\Psi_{\pm}(x)$, $\Psi_{\pm}(x) = \Psi(x) \pm \Psi(-x)$. This explains the name of operator R. One can note that if we write realization (2.4) in the Schrödinger representation just in the case of non-deformed ($\nu = 0$) Heisenberg algebra, we shall reveal a hidden nonlocal nature of the reflection operator, $R = \sin H_0$, $H_0 = \frac{1}{2}(x^2 - d^2/dx^2)$. Therefore, the reflection operator has a nature similar to the nonlocal nature of the Klein operator [1].

One can get the realization of the R-deformed Heisenberg algebra in terms of non-deformed algebra with creation-annihilation operators $b^{\pm}$ obeying the commutation relation $[b^-, b^+] = 1$. For the purpose, one represents the operators $a^{\pm}$ as $a^- = F(N_b)b^-$, $a^+ = (a^-)^{\dagger} = b^+F(N_b)$ with $F = F^{\dagger}$ being a function of the number operator $N_b = b^+b^-$. Let us substitute these expressions for a^+ and a^- and $R = (-1)^{N_b}$ into the first relation from (2.1), and act on the complete set of orthonormal states $|n\rangle_b \equiv (n!)^{-1/2}(b^+)^n|0\rangle = |n\rangle$, $N_b|n\rangle_b = n|n\rangle_b$, where $|n\rangle$, $n = 0, 1, \ldots$, are the Fock space states of deformed algebra. As a result, we arrive at the sought for realization of deformed creation-annihilation operators in terms of non-deformed ones,

$$a^- = F(N_b)b^-, \quad a^+ = b^+F(N_b), \tag{2.5}$$

$$F(N_b) = \sqrt{1 + \frac{\nu}{2(N_b+1)}(1 + (-1)^{N_b})}, \quad \nu > -1. \tag{2.6}$$

Function (2.6) takes zero values if we put $\nu = -(2p+1)$, $p = 1, 2, \ldots$. This indicates [14] that at these special values of the deformation parameter algebra (2.1) has finite-dimensional representations. Then, using eq. (2.2), one finds that for $\nu = -(2p+1)$, $\quad p = 1, 2, \ldots$, the relation $\langle\langle m|n\rangle\rangle = 0$, $|n\rangle\rangle \equiv (a^+)^n|0\rangle$, takes place for $n \geq 2p+1$ and arbitrary m. This means, in turn, that the relations $(a^+)^{2p+1} = (a^-)^{2p+1} = 0$ are valid in this case. These latter relations specify finite-dimensional representations of the R-deformed Heisenberg algebra. Since in such representations for any $p = 1, 2, \ldots$, there are the states with negative norm (see eq. (2.2)), it means that these finite-dimensional Fock space representations are non-unitary.

3 R-paragrassmann algebra

Let us consider the revealed finite-dimensional representations in more detail. We have arrived at the nilpotent algebra

$$[a^-, a^+] = 1 - (2p+1)R, \quad (a^{\pm})^{2p+1} = 0, \quad p = 1, 2, \ldots, \tag{3.1}$$

$$\{a^{\pm}, R\} = 0, \quad R^2 = 1. \tag{3.2}$$

One can interpret a^+ as a paragrassmann variable θ, $\theta^{2p+1} = 0$, and in this case a^- can be considered as a differentiation operator [10]. Therefore, the algebra (3.1), (3.2) is a

paragrassmann algebra of order $2p+1$ [10] with a special differentiation operator whose action can be defined by relation (2.2). We shall call it the R-paragrassmann algebra. Here, in addition to universal representation (2.3), (2.4), one has also the normal ordered representation for the operator R,

$$R = \sum_{n=0}^{2p} f_n a^{+n} a^{-n}, \tag{3.3}$$

with finite recursive relations defining coefficients f_n,

$$2f_{n-1} + [n]_\nu f_n - (2p+1) \sum_{i=0}^{[n/2]-1} f_{2i+1} f_{n-(2i+1)} = 0, \quad n = 1, \ldots, 2p,$$

where $f_0 = 1$ and $[n/2]$ is an integer part of $n/2$.

As a consequence of eqs. (3.1), (3.2), we have the relations $(1-R)a^{+2p} = (1-R)a^{-2p} = 0$. They are equivalent to the nilpotency conditions $a^{\pm(2p+1)} = 0$. Besides, here operators $a^\pm$ satisfy the relation

$$a^+ a^{-2p} + a^- a^+ a^{-(2p-1)} + \ldots a^{-(2p-1)} a^+ a^- + a^{-2p} a^+ = 0 \tag{3.4}$$

and corresponding conjugate relation. Relations of form (3.4) take place in parasupersymmetric quantum mechanics [15].

As it has been mentioned above, finite-dimensional Fock space representations of R-deformed Heisenberg algebra (2.1) contain the states with negative norm. One may introduce the normalized states as $|n\rangle = |\langle\langle n|n\rangle\rangle|^{-1/2}|n\rangle\rangle$. They define the metric operator $\eta = \eta^\dagger$, $\eta^2 = 1$, whose matrix elements are $||\eta||_{mn} = ||\langle m|n\rangle|| = diag(1, -1, -1, +1, +1, -1, -1, \ldots, (-1)^{p-1}, (-1)^{p-1}, (-1)^p, (-1)^p)$. With this metric operator, the indefinite scalar product is given by the relation $(\Psi_1, \Psi_2) = \langle \Psi_1|\eta\Psi_2\rangle = \Psi^*_{1n}\eta_{nm}\Psi_{2m}$, where $\Psi_n = \langle n|\Psi\rangle$. The operators a^+ and a^- can be represented by the matrices $(a^+)_{mn} = A_n \delta_{m-1,n}$, $(a^-)_{mn} = B_m \delta_{m+1,n}$, with $A_{2k+1} = -B_{2k+1} = \sqrt{2(p-k)}$, $k = 0, 1, \ldots, p-1$, $A_{2k} = B_{2k} = \sqrt{2k}$, $k = 1, \ldots, p$. They satisfy the relation $(a^-)^\dagger = \eta a^+ \eta$, and, as a consequence, are mutually conjugate operators with respect to this scalar product, $(\Psi_1, a^-\Psi_2)^* = (\Psi_2, a^+\Psi_1)$. The reflection operator has here the diagonal form $R = diag(+1, -1, +1, \ldots, -1, +1)$.

Below we shall reveal the 'physical explanation' of non-unitarity of finite-dimensional representations of the R-deformed Heisenberg algebra in which a^+ and a^- are interpreted as mutually conjugate operators. On the other hand, one can define hermitian conjugate operators $f^+ = a^+$, $f^- = a^- R$, in terms of which R-paragrassmann algebra (3.1), (3.2) is rewritten equivalently as

$$\{f^+, f^-\} = (2p+1) - R, \quad \{R, f^\pm\} = 0, \quad R^2 = 1, \quad (f^\pm)^{2p+1} = 0, \quad p = 1, 2, \ldots. \tag{3.5}$$

With these operators one could work in a Hilbert space with positive definite scalar product $(\Psi_1, \Psi_2) = \Psi^*_{1n}\Psi_{2n}$, considering a^+ and a^- as not basic operators. However, due to concrete physical applications to be considered in what follows, here we shall work in terms of operators $a^\pm$ using the corresponding indefinite scalar product. The described possibility of employing hermitian conjugate operators f^+ and f^- will be discussed in last section.

4 Guons, fermions and q-deformed Heisenberg algebra

Let us suppose that $\nu \neq 1$, and define the operators $c^- = a^- G_\nu^{-1/2}(R)$, $c^+ = G_\nu^{-1/2}(R)a^+$, $G_\nu(R) = |1 - \nu R|$, where for the moment we suppose that $R = (-1)^N$ with N given by eq. (2.3). These operators anticommute with reflection operator, $\{R, c^\pm\} = 0$, and satisfy the commutation relation $c^-c^+ - G_\nu(R)G_\nu^{-1}(-R)c^+c^- = sign(1 + \nu R)$, where $sign\, x$ is $+1$ for $x > 0$ and -1 for $x < 0$. The operator $G_\nu(R)$ is reduced to $G_\nu(R) = 1 - \nu R$ for $-1 < \nu < 1$; for two other cases we have $G_\nu(R) = \nu - R$, $\nu > 1$, and $G_\nu(R) = R - (2p+1)$, $\nu = -(2p+1)$. As a result, commutation relation is represented in first case as

$$c^-c^+ - g_\nu c^+c^- = 1, \quad g_\nu = (1-\nu)^R(1+\nu)^{-R}, \quad -1 < \nu < 1, \tag{4.1}$$

whereas in two other cases it is reduced to

$$c^-c^+ - g_\nu c^+c^- = R, \tag{4.2}$$

where $g_\nu = (\nu - 1)^R(1+\nu)^{-R}$ for $\nu > 1$ and $g_\nu = p^R(1+p)^{-R}$ for $\nu = -(2p+1)$. In the case corresponding to finite-dimensional representations the final form (4.2) has been obtained via additional changing $R \to -R$. In all three cases operator-valued function g_ν satisfies the relation $g_\nu c^\pm = c^\pm g_\nu^{-1}$. The deformed algebra of form (4.1) was introduced in ref. [11] in the context of generalized statistics. The algebra (4.2) represents some modification of (4.1).

The corresponding number operator $N = N(c^+, c^-)$ is given by

$$N = -\frac{\alpha}{2} + \frac{1}{2}\sqrt{|1-\nu^2|(2c^+c^- - \beta)(2c^-c^+ - \beta) + 1},$$

where $\alpha = -\nu$, $\beta = 1$ in the case $-1 < \nu < 1$, and $\alpha = -\nu + \nu^2 - 1$, $\beta = |\nu|$ for two other cases. Implying in relations (4.1), (4.2) that $R = (-1)^{N(c^+,c^-)}$, one can represent them in a closed form containing only creation-annihilation operators $c^\pm$.

Let us take a limit $\nu \to \infty$ for the case $\nu > 1$ and $p \to \infty$ for $\nu = -(2p+1)$ proceeding from relation (4.2). Both cases lead to the algebra

$$c^-c^+ - c^+c^- = R, \quad \{R, c^\pm\} = 0, \quad R^2 = 1. \tag{4.3}$$

Considering the Fock space representation defined by relations $c^-|0\rangle = 0$, $R|0\rangle = |0\rangle$, $\langle 0|0\rangle = 1$, one gets the relations $\langle 1|1\rangle = 1$, $\langle 0|1\rangle = 0$, and $\langle m|n\rangle = 0$ for any $m \geq 2$ or $n \geq 2$, where $|n\rangle = (c^+)^n|0\rangle$. It means that the (anti)commutation relations (4.3) have two-dimensional irreducible representation, in which $(c^\pm)^2 = 0$. In this case the operator R is realized as $R = 1 - 2c^+c^-$, that reduces commutation relations (4.3) to the standard fermionic anticommutation relations, $c^+c^- + c^+c^- = 1$, $c^{+2} = c^{-2} = 0$. Therefore, fermionic algebra can be obtained from the guon-like form of the R-deformed Heisenberg algebra in the limit $|\nu| \to +\infty$.

The substitution of operators $c^\pm$ into bosonic realization (2.5), (2.6) gives for $\nu \to \infty$ the well known realization of fermionic operators in terms of bosonic operators $b^\pm$ [16, 6, 7]:

$$c^- = \frac{\Pi_+}{\sqrt{N+1}}b^-, \quad c^+ = (c^-)^\dagger. \tag{4.4}$$

Here Π_+ and supplementary operator Π_-, $\Pi_\pm = \frac{1}{2}(1 \pm R)$, are projector operators, $\Pi_\pm^2 = \Pi_\pm$, $\Pi_+\Pi_- = 0$, $\Pi_+ + \Pi_- = 1$, and in eq. (4.4) we mean that $R = (-1)^{N_b}$. Due to relations $\Pi_\pm b^- = b^-\Pi_\mp$, $\Pi_\pm b^+ = b^+\Pi_\mp$, operators (4.4) satisfy the standard fermionic anticommutation relations.

This bosonization construction for fermions gives us a hint for realization of R-paragrassmann algebra in terms of non-deformed creation-annihilation operators $b^\pm$. Indeed, the operators $a^\pm$ satisfying algebra (3.1), (3.2) can be realized as follows:

$$a^- = \varphi_p(N_b)F_p(N_b)b^-, \quad a^+ = b^+F_p(N_b)\varphi_p(N_b), \tag{4.5}$$

where instead of projector operator $\Pi_+ = \sin(\frac{\pi}{2}(N_b+1))$, we have

$$\varphi_p(N_b) = \frac{\sin\left(\frac{\pi}{2p+1}(1+N_b)\right)}{\sin\left(\frac{\pi}{2p+1}(1+\mathcal{N}_p)\right)} \tag{4.6}$$

with operator $\mathcal{N}_p = \mathcal{N}_p(N_b)$,

$$\mathcal{N}_p(N_b) = p + \frac{1}{2} - \left| N_b - \left(p + \frac{1}{2} + (2p+1)\left[\frac{N_b}{2p+1}\right]\right)\right|. \tag{4.7}$$

Here $[X]$ is an integer part of X, and operator $F_p(N_b)$ is given by

$$F_p(N_b) = i^{N_b+1}\sqrt{\left|1 - \frac{2p+1}{2(N_b+1)}\left(1+(-1)^{N_b}\right)\right|}.$$

Operator φ_p has the properties $|\varphi_p(n)| = 1$, $n \neq 2p \; mod(2p+1)$, $\varphi_p(2p + k(2p+1)) = 0$, $k \in Z$, and so, $\varphi_p(N_b)\varphi_p(N_b+1)\ldots\varphi_p(N_b+2p) = 0$. Due to the latter property and relations $G(N_b)b^\pm = b^\pm G(N_b \pm 1)$ being valid for any function $G(N_b)$, one concludes that operators (4.5) satisfy relations $(a^\pm)^{2p+1} = 0$. They are odd operators, $Ra^\pm = -a^\pm R$, $R = (-1)^{N_b}$, and satisfy R-deformed commutation relations (2.1). Since the relation $(a^-)^\dagger\eta = \eta a^+$ takes place with $\eta = \eta^\dagger = (-1)^{[(N_b+1)/2]}$, the operators a^- and a^+ are mutually conjugate with respect to the indefinite scalar product $(\Psi_1, \Psi_2) = \langle\Psi_1|\eta\Psi_2\rangle$. The obtained bosonized representation corresponds to finite-dimensional matrix representation of the R-deformed Heisenberg algebra described in the previous section.

Special form of fermionic algebra (4.3) can be generalized into the algebra related to the q-deformed oscillator. To realize such a generalization, we note that since $R^2 = 1$, R is a phase operator. Then commutation relations (4.3) (R-algebra) can be generalized into the P-algebra,

$$[a, \bar{a}] = P, \tag{4.8}$$

where P is a phase operator with properties generalizing the corresponding properties of operator R,

$$P^p = 1, \quad Pa = qaP, \quad P\bar{a} = q^{-1}\bar{a}P, \quad q = e^{-i\frac{2\pi}{p}}, \quad p = 2, 3, \ldots. \tag{4.9}$$

Using these relations, one finds that the operators a^p and $\bar{a}^p$ commute with operators a, $\bar{a}$ and P. In an irreducible representation they are reduced to some constants. Assuming the

existence of the vacuum state $|0\rangle$, $a|0\rangle = 0$, we find that in Fock space representation of algebra (4.8), (4.9), there are the relations $a^p = \bar{a}^p = 0$. Multiplying relation (4.8) from the left by the operator $P^{-1} = P^{p-1}$, we represent it in the form of Lie-admissible algebra [11], $aT\bar{a} - \bar{a}Sa = 1$, $T = q^{-1}P^{-1}$, $S = qP^{-1}$. Defining new creation-annihilation operators, $c = q^{-1/2}aP^{-1/2}$, $\bar{c} = q^{-1/2}P^{-1/2}\bar{a}$, one gets finally the q-deformed Heisenberg algebra $c\bar{c} - q\bar{c}c = 1$, with deformation parameter q being the primitive root of unity.

5 OSp(2|2) supersymmetry

The R-deformed Heisenberg algebra gives a possibility to realize OSp(2|2) supersymmetry. As a result we can get unitary inifinite-dimensional half-bounded representations of $sl(2,R)$, $sl(2,R) \subset osp(1|2) \subset osp(2|2)$, and its non-unitary finite-dimensional representations.

In terms of generators of algebra (2.1), the generators of $osp(2|2)$ superalgebra can be realized as follows. The even generators J_0, $J_\pm = J_1 \pm iJ_2$ and Δ are given by relations

$$J_0 = \frac{1}{4}\{a^-, a^+\}, \quad J_\pm = \frac{1}{2}(a^\pm)^2, \quad \Delta = -\frac{1}{2}(R + \nu). \tag{5.1}$$

They satisfy $sl(2,R) \times u(1)$ algebra,

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_-, J_+] = 2J_0, \quad [\Delta, J_0] = [\Delta, J_\pm] = 0.$$

Odd generators $Q^\pm$, $S^\pm$ are represented as $Q^+ = a^+\Pi_-$, $Q^- = a^-\Pi_+$, $S^+ = a^+\Pi_+$, $S^- = a^-\Pi_-$, where $\Pi_\pm$ are the projectors, $\Pi_\pm = \frac{1}{2}(1 \pm R)$. These operators satisfy the following anticommutation relations:

$$Q^{\pm 2} = S^{\pm 2} = 0, \quad \{S^+, Q^-\} = \{S^-, Q^+\} = 0, \tag{5.2}$$

$$\{Q^+, Q^-\} = 2J_0 + \Delta, \quad \{S^+, S^-\} = 2J_0 - \Delta, \quad \{S^+, Q^+\} = J_+, \quad \{S^-, Q^-\} = J_-. \tag{5.3}$$

Nontrivial commutators between even and odd generators are

$$[J_+, Q^-] = -S^+, \quad [J_+, S^-] = -Q^+, \quad [J_-, Q^+] = S^-, \quad [J_-, S^+] = Q^-, \tag{5.4}$$

$$[J_0, Q^\pm] = \pm\frac{1}{2}Q^\pm, \quad [J_0, S^\pm] = \pm\frac{1}{2}S^\pm, \quad [\Delta, Q^\pm] = \pm Q^\pm, \quad [\Delta, S^\pm] = \pm S^\pm. \tag{5.5}$$

In the case $\nu > -1$, the generators J_0, $J_\pm$ give the direct sum of half-bounded infinite-dimensional unitary representations $D^+_{\alpha_+}$ and $D^+_{\alpha_-}$ of $sl(2,R)$, being representations of the so called discrete series. Here $\alpha_+ = \frac{1}{4}(1+\nu) > 0$, and $\alpha_- = \alpha_+ + \frac{1}{2}$, and these representations are realized on the subspaces spanned by $|2n\rangle$ and $|2n+1\rangle$, $n = 0, 1, \ldots$, where the corresponding Casimir operator of $sl(2,R)$, $C = -J_0^2 + \frac{1}{2}\{J_+, J_-\}$, takes the values $C = -\alpha_+(\alpha_+ - 1)$ and $C = -\alpha_-(\alpha_- - 1)$, and J_0 has the spectra $j_0 = \alpha_+ + n$ and $j_0 = \alpha_- + n$, respectively [7, 8]. In the case of the revealed finite-dimensional representations of the R-deformed Heisenberg algebra, one finds that the generators J_0, $J_\pm$ give a direct sum of two non-unitary $(p+1)-$ and $p-$dimensional irreducible representations characterized by the values of the Casimir operator $C = -j_\pm(j_\pm + 1)$ with $j_+ = p/2$ and $j_- = (p-1)/2$. These representations are realized on the subspaces of even and odd states, $|m\rangle_+ = a^{+2m}|0\rangle$, $m = 0, 1 \ldots, p$, $|m\rangle_- =$

$a^{+(2m+1)}|0\rangle$, $m=0,\dots,p-1$, where J_0 has the spectra $j_0=(-j_+,-j_++1,\dots,j_+)$ and $j_0=(-j_-,-j_-+1,\dots,j_-)$. In other words, finite-dimensional representations of the R-deformed Heisenberg algebra give finite-dimensional representations of $(2+1)$-dimensional Lorentz group. As it was noted in section 3, the appearance of indefinite scalar product in the case of finite-dimensional representations means that such representations are non-unitary. As we have seen, these representations are the direct sum of finite-dimensional representations of (2+1)-dimensional Lorentz group, and since finite-dimensional representations of this group are non-unitary (see, e.g., ref. [17]), we have here a 'physical explanation' for non-unitarity of finite-dimensional representations of the R-deformed Heisenberg algebra with operators a^+ and a^- to be mutually conjugate.

In the simplest cases given by $p=1$ and $p=2$, the corresponding metric operator in two-dimensional even ($p=1$, $j_+=1/2$) and odd ($p=2$, $j_-=1/2$) subspaces coincides up to a c-number factor with the operator J_0 being restricted to the corresponding subspaces. As a result, the indefinite scalar product on these subspaces is the Dirac scalar product. In the case of 3-dimensional vector representations corresponding to $j_+=1$, $p=2$ and $j_-=1$, $p=3$, the metric operator and generators J_μ, $\mu=0,1,2$, being restricted to the corresponding even and odd subspaces, can be reduced by appropriate unitary transformation to the standard form of the vector realization with $(J_\mu)^\nu{}_\lambda=-i\epsilon^\nu{}_{\mu\lambda}$ and $\eta_{\mu\nu}=diag(-,+,+)$ [17].

As we have seen, the generators of $sl(2,R)$ algebra act reducibly in the cases of infinite-dimensional and finite-dimensional representations of algebra (2.1). On the other hand, these generators together with operators $a^\pm=Q^\pm+S^\pm$ give irreducible realization of $osp(1|2)$ generators with corresponding Casimir operator [2, 5, 6, 7] $\mathcal{C}=J_\mu J^\mu-\frac{1}{8}[a^-,a^+]$ taking the fixed value $\mathcal{C}=\frac{1}{16}(1-\nu^2)$.

In conclusion of this section we note that relations (5.2) and (5.5) mean that the pair of odd generators Q^+ and Q^- together with even generator $H_+=2J_0+\Delta$ form $s(2)$ superalgebra, $Q^{\pm 2}=0$, $\{Q^+,Q^-\}=H_+$, $[Q^\pm,H_+]=0$, whereas operators S^+ and S^- are odd generators of $s(2)$ superalgebra with even generator $H_-=2J_0-\Delta$.

6 Outlook and concluding remarks

The constructed guon-like algebra of the form (4.1), (4.2) contains the operator-valued function g_ν. But unlike the original guon algebra [11], here $g_\nu^2\neq 1$ and $[g_\nu,c^\pm]\neq 0$. The condition of the form $[g,c^\pm]=0$ appeared in [11] from the requirement of micro causality under assumption that observables should be bilinear in fields or in creation-annihilation operators. On the other hand, it is known that in the field-theoretical anyonic constructions involving the Chern-Simons gauge field, there are observables (e.g., total angular momentum operator) which are not bilinear in creation-annihiliation operators [18]. Moreover, the gauge-invariant fields carrying fractional spin and statistics themselves turn out to be nonlocal operators [19] being decomposable in some infinite series in degrees of creation-annihilation operators of the initial matter field. It seems that the guon-like algebra appeared here could find some applications in the theory of anyons.

The revealed finite-dimensional representations of the R-deformed Heisenberg algebra and their relationship to representations of $(2+1)$-dimensional Lorentz group can be used for the construction of universal minimal spinor set of linear differential equations describing,

on one hand, ordinary integer and half-integer spin fields and, on other hand, fractional spin fields in $2+1$ dimensions. Moreover, it is natural to try to apply these representations for constructing $(2+1)$-dimensional supersymmetric field systems since, as it was shown, any $(2p+1)$-dimensional representation of the R-deformed Heisenberg algebra carries the direct sum of spin-j, $j=p/2$, and spin-$(j-1/2)$ representations of $(2+1)$-dimensional Lorentz group. For the purpose, let us consider the simplest possible nontrivial case corresponding to the choice of 5-dimensional representation of the R-deformed Heisenberg algebra with $\nu=-5$ $(p=2)$, and construct the operators

$$\mathcal{D}_\alpha = \left(\frac{1}{2}-R\right)\mathcal{P}_\alpha - \mathcal{J}_\alpha + \frac{1}{2}\epsilon m \mathcal{L}_\alpha, \quad \epsilon=+,-.$$

Here $\alpha=1,2$, m is a mass parameter, $\mathcal{P}_\alpha = -i(\gamma^\mu\partial_\mu)_\alpha{}^\beta\mathcal{L}_\beta$, $\partial_\mu=\partial/\partial x^\mu$, x^μ are external space-time coordinates independent from $a^\pm$, γ_μ is the set of (2+1)-dimensional γ-matrices taken in the Majorana representation, $(\gamma^0)_\alpha{}^\beta=-(\sigma^2)_\alpha{}^\beta$, $(\gamma^1)_\alpha{}^\beta=i(\sigma^1)_\alpha{}^\beta$, $(\gamma^2)_\alpha{}^\beta=i(\sigma^3)_\alpha{}^\beta$, $\mathcal{L}_1=\frac{1}{\sqrt{2}}(a^+ + a^-)$, $\mathcal{L}_2=\frac{i}{\sqrt{2}}(a^+ - a^-)$, and $\mathcal{J}_\alpha = \mathcal{L}_\beta\epsilon_{\mu\nu\lambda}\partial^\mu J^\nu(\gamma^\lambda)_\alpha{}^\beta$ with J_μ given by eq. (5.1). Operators $\mathcal{L}_\alpha$, $\mathcal{J}_\alpha$ and $\mathcal{P}_\alpha$ are spinor operators with respect to the action of the total angular momentum vector operator, $M_\mu = i\epsilon_{\mu\nu\lambda}x^\nu\partial^\lambda + J_\mu$, $[M_\mu, M_\nu] = -i\epsilon_{\mu\nu\lambda}M^\lambda$, $[M_\mu,\mathcal{L}_\alpha]=\frac{1}{2}(\gamma_\mu)_\alpha{}^\beta\mathcal{L}_\beta$ etc., whereas the reflection operator R is a scalar, $[M_\mu, R]=0$. These properties of the operators are, in fact, the consequence of the $osp(1|2)$ superalgebra generated by the operators J_μ and $a^\pm$, which has been discussed in the previous section. As a result, operator $\mathcal{D}_\alpha$ is (2+1)-dimensional translation-invariant spinor operator. One can consider the set of linear (in ∂_μ) differential field equations

$$\mathcal{D}_\alpha\Psi(x)=0 \tag{6.1}$$

having in mind that $\Psi(x)$ is a 5-component field, which with respect to (2+1)-dimensional Lorentz group is transformed as $\Psi(x)\rightarrow\Psi'(x')=\exp(iM_\mu\omega^\mu)\Psi(x)$, where ω^μ are the transformation parameters. Therefore, eq. (6.1) is the covariant (spinor) set of (2+1)-dimensional field equations. One can find that the field $\Psi(x)$ satisfying equations (6.1) is decomposed into the sum of fields $\Psi_\pm=\Pi_\pm\Psi$, $\Psi=\Psi_+ + \Psi_-$, carrying spins $s_+=-\epsilon$ and $s_-=\frac{1}{2}s_+$, respectively. Field Ψ_- is a 2-component Dirac field, whereas 3-component field Ψ_+ is, in fact, topologically massive Jackiw-Templeton-Deser-Schonfeld vector field [20, 17]. Both fields have the same mass m, and, therefore, spinor set of equations (6.1) describes a supermultiplet of (2+1)-dimensional massive fields. The spinor supercharge operator generating the corresponding supertransformations is

$$\mathcal{Q}_\alpha = \epsilon m\mathcal{L}_\alpha + R\mathcal{P}_\alpha. \tag{6.2}$$

It anticommutes with $\mathcal{D}_\alpha$ on mass shell, i.e. on the surface of equations (6.1), $\{\mathcal{Q}_\alpha,\mathcal{D}_\beta\}\approx 0$, and satisfies the relations $\{\mathcal{Q}_\alpha,\mathcal{Q}_\beta\}\approx -16\epsilon m(\gamma_\mu\partial^\mu)_{\alpha\beta}$, $[\partial_\mu,\mathcal{Q}_\alpha]=0$. Now, one can introduce a field $\Phi(x)$ carrying arbitrary (fixed) infinite- or finite-dimensional representation of the R-deformed Heisenberg algebra, and consider another spinor set of equations,

$$\mathcal{Q}_\alpha\Phi(x)=0. \tag{6.3}$$

Here operator $\mathcal{Q}_\alpha$ (6.2) is generalized to the case of the corresponding representation. Solution of eq. (6.3) is decomposable into the trivial field $\Psi_-(x)=\Pi_-\Phi(x)=0$ and field

$\Phi_+(x) = \Pi_+\Phi(x)$ carrying irreducible representation of the (2+1)-dimensional Poincaré group characterized by mass m and spin $s_+ = \epsilon\frac{1}{4}(1+\nu)$. Therefore, spin of nontrivial field Φ_+ is defined by the value of deformation parameter ν, and one concludes that the spinor set of equations (6.3) is the above-mentioned universal set of linear differential equations giving some link between fractional spin fields (anyons) in the case of choosing $\nu > -1$ [7, 8], and ordinary (2+1)-dimensional integer and half-integer spin fields in the case $\nu = -(2p+1)$. The described (2+1)-dimensional supersymmetry as well as the universal spinor set of linear differential field equations will be considered in detail elsewhere [21]. We only note here that the spinor sets of equations (6.1) and (6.3) are analogous to (3+1)-dimensional Dirac positive-energy equations [22] in the sense that they represent by themselves the covariant (spinor) sets of equations imposed on one multi-component field.

In conclusion we note that in terms of hermitian conjugate operators $f^+ = a^+$ and $f^- = a^-R$ satisfying anticommutation relations (3.5), the described finite-dimensional representations of the R-deformed Heisenberg algebra supply us with some special deformation of parafermionic algebra of order $2p$ with internal Z_2 grading structure [21]. Recently it was shown [23] that new variants of parasupersymmetry can be constructed with the help of finite-dimensional representations of the q-deformed Heisenberg algebra. It turns out that physical properties of such new variants can be different from the properties of the parasupersymmetry realized in terms of the standard parafermionic generators [15]. Therefore, the revealed finite-dimensional representations of the R-deformed Heisenberg algebra may also be interesting from the point of view of constructing parasupersymmetric systems. Perhaps, there the intrinsic Z_2-grading structure of the corresponding deformed parafermionic algebra could find physically interesting consequences.

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